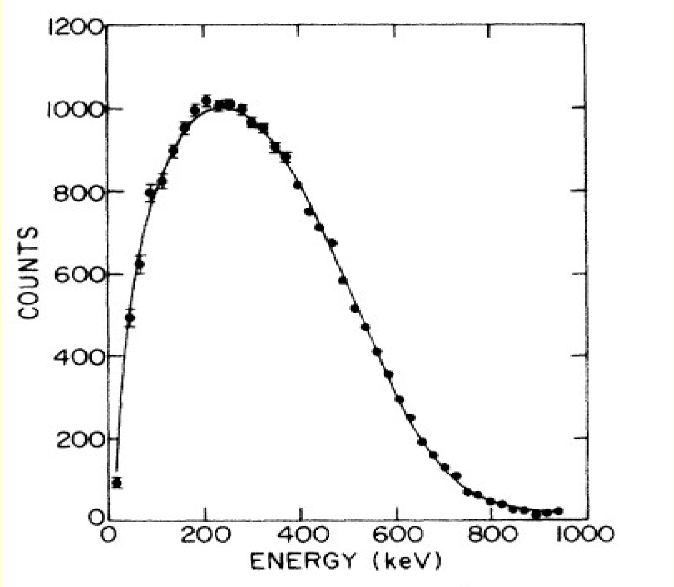
$$n \rightarrow p + e^- + \bar{\nu}_e$$

(and related processes...)



Goals:

- understand the shape of the energy spectrum
- total decay rate sheds light on the underlying weak interaction mechanism

Starting point: "Fermi's Golden Rule" again! *(lecture 6)*

$$\boxed{\lambda_{if} = \frac{2\pi}{\hbar} \left| M_{if} \right|^2 \rho_f}$$

(transitions / sec)

matrix element

$$\boxed{M_{if} = \int \psi_f^* \, V(\vec{r}) \, \psi_i \, d^3 r}$$

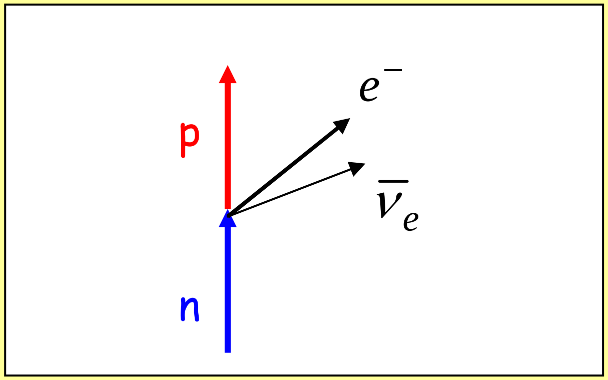
$$\boxed{\rho_f = dn / dE_f}$$

density of states

Matrix element M_{if}?

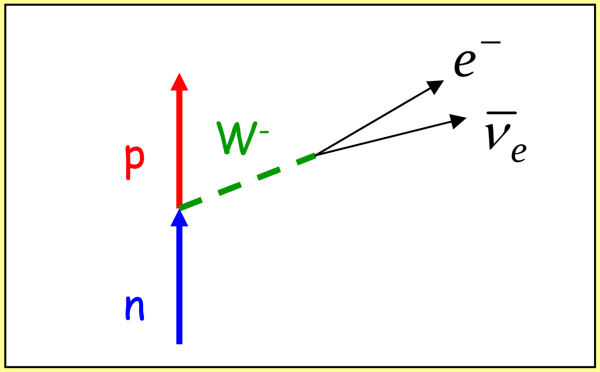
Simplest model is to take a pointlike interaction with an overall energy scale "G":

(Fermi, 1934 – almost right!)



$$M_{if} = G \int \psi^*_{p,f}(\vec{r})\, \phi^*_e(\vec{r})\, \phi^*_\nu(\vec{r})\, \psi_{n,i}(\vec{r})\; d^3r$$

(the interaction is proportional to the wavefunction overlap of initial and final state particles at the same point in space)



Standard Model description: an extended interaction, but the range is only about 0.002 fm which is just about zero!

The Standard Model can 'predict' the value of G in terms of model parameters, whereas in Fermi's theory, it remains to be determined from experiment.

Spin considerations: electron and neutrino

There are two possibilities for the angular momentum coupling of the two leptons!

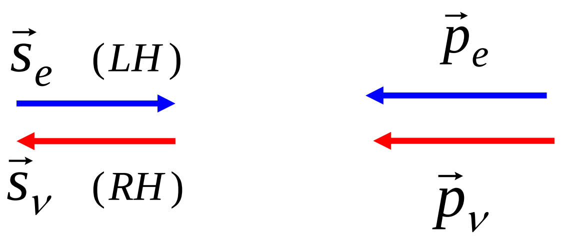
$$\vec{s}_e + \vec{s}_\nu = \vec{S}_{tot}, \quad \Rightarrow \quad S = 0 \quad or \quad 1$$

For neutron decay: $n \rightarrow p + e^- + \bar{\nu}_e$

Angular momentum: $\vec{\frac{1}{2}} = \vec{\frac{1}{2}} + \vec{S}$ **both can contribute to neutron decay!**

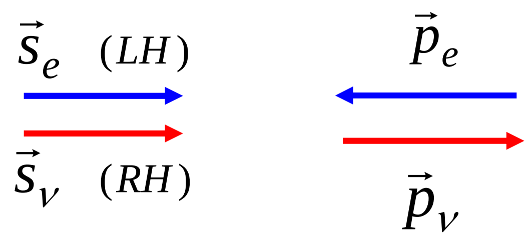
Subtle point: because the leptons are emitted with a definite **helicity**
we can deduce a correlation between their directions of motion in the two cases:

Fermi Decay, S = 0:



e- and ν travel in the **same** direction

Gamow-Teller decay, S = 1:

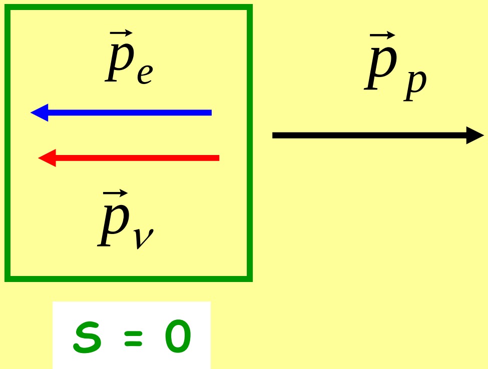


e- and ν travel in **opposite** directions

More on spin correlations:

Fermi Decay, S = 0: $n \rightarrow p + e^{-} + \overline{\nu}_e$

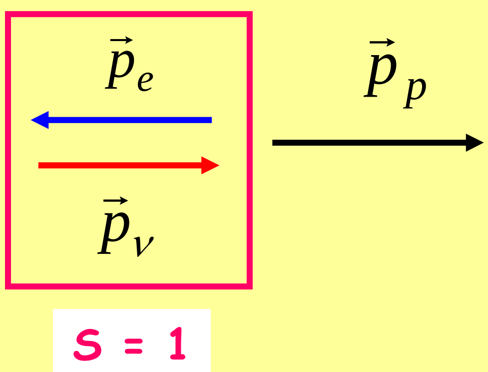
Leptons travel in the **same** direction::



Recoiling proton spin is in the **same** direction as the initial neutron spin.

Gamow – Teller Decay, S = 1:

Leptons travel in the **opposite** direction:



Recoiling proton spin is in the **opposite** direction as the initial neutron spin, i.e. a "spin flip"

As before, assume a pointlike interaction, but allow for different coupling constants for the Fermi (F) and Gamow-Teller (GT) cases.

Fermi case, S = 0: **(coupling constant: "G_V" because the operator transforms like a space vector.)**

$$M_{if} \equiv M_F = G_V \int \psi^*_{p,f}(\vec{r})\, \phi^*_e(\vec{r})\, \phi^*_\nu(\vec{r})\, \psi_{n,i}(\vec{r}) \;\; d^3r$$

Gamow–Teller, S = 1: **(coupling constant: "G_A" because the operator transforms like an axial vector, i.e. like angular momentum.)**

$$M_{if} \equiv M_{GT} = G_A \int \psi^*_{p,f}(\vec{r})\, \phi^*_e(\vec{r})\, \phi^*_\nu(\vec{r})\, \psi_{n,i}(\vec{r}) \;\; d^3r$$

Experimentally, the coupling constants are very similar:

$$G_A / G_V = -1.25$$

(These are evaluated by comparing different nuclear beta decay transitions, where angular momentum conservation restricts the total lepton spin states that contribute)

Transition rate: $n \rightarrow p + e^- + \bar{\nu}_e$ (can proceed with S = 0 or 1)

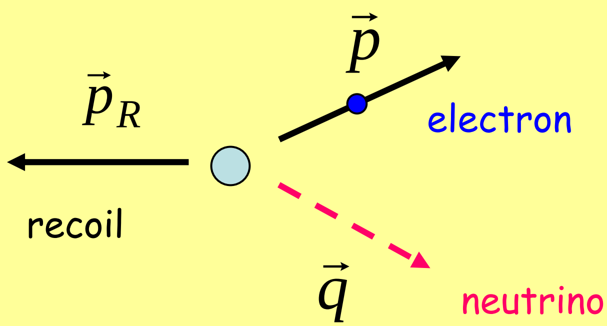
for the neutron:

$$\boxed{\lambda_{if} = \frac{2\pi}{\hbar} \left| M_{if} \right|^2 \rho_f \quad \sim \; (G_V^2 \; + 3\, G_A^2)}$$

since there are 3 times as many ways for the leptons to be emitted with S = 1 (m_s= 1, 0, -1) as with S = 0.

For now, let us work out a generic matrix element, since the expressions are the same for both apart from the coupling constants:

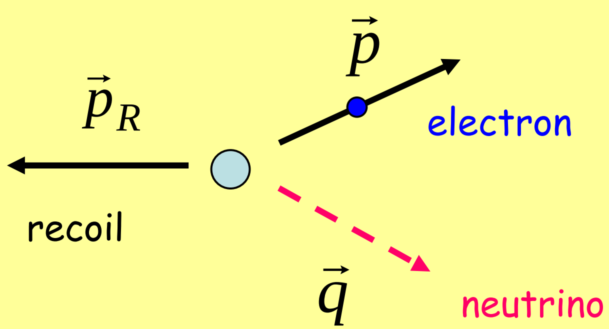
$$M_{if} \; = \; G \int \psi^*_{p,f}(\vec{r})\, \phi^*_e(\vec{r})\, \phi^*_\nu(\vec{r})\, \psi_{n,i}(\vec{r}) \;\; d^3 r$$



Electron and neutrino are represented by plane wave functions of definite momentum:

$$\phi_e(\vec{r}) \; = \; \frac{e^{\,i\vec{p}\bullet\vec{r}\,/\,\hbar}}{\sqrt{V}} \; , \; etc.$$

Matrix element:



$$M_{if} = G \int \psi^*_{p,f}(\vec{r})\, \phi^*_e(\vec{r})\, \phi^*_\nu(\vec{r})\, \psi_{n,i}(\vec{r})\ d^3 r$$

$$\vec{p}_R = -(\vec{p} + \vec{q})$$

$$\phi^*_e\ \phi^*_\nu = \left(\frac{1}{V}\right) e^{-i\,(\vec{p}+\vec{q})\bullet\vec{r}/\hbar} = \left(\frac{1}{V}\right) e^{i\ \vec{p}_R \bullet \vec{r}/\hbar}$$

The integral for M_{if} extends over all space regions for which the **nucleon wave functions (n,p) are non-zero: R_{max} ~ 1 fm** (in nuclei, use R ~ 1.2 $A^{1/3}$ fm) ...

But, the **recoil momentum p_R is no larger than the Q-value** for the reaction, ~ MeV ...

$$\vec{p}_R \bullet \vec{r}/\hbar \leq \frac{Q\, R_{max}}{\hbar c} \sim \frac{1 \quad \text{MeV.fm}}{197 \quad \text{MeV.fm}} << 1 \Longrightarrow \boxed{\phi^*_e\ \phi^*_\nu \cong \frac{1}{V}}$$

This is a great simplification: the lepton wave functions are just a constant over the region of space that matters to calculate the matrix element ☺

Summary so far for the transition rate calculation:

$$\lambda_{if} = \frac{2\pi}{\hbar} \left| M_{if} \right|^2 \rho_f = \frac{2\pi}{\hbar} \frac{G^2}{V^2} \left| \underbrace{\int \psi^*_{f,p}\, \psi_{i,n}\, d^3r} \right|^2 \rho_f$$

The remaining integral is referred to as the nuclear matrix element:

$$M_{nuclear} \equiv \int \psi^*_{f,p}(\vec{r})\ \psi_{i,n}(\vec{r})\, d^3r$$

When beta decay occurs in a nucleus, the initial and final wave functions of the proton and neutron need not be exactly the same, so in general:

$$\left| M_{nuclear} \right| \leq 1$$

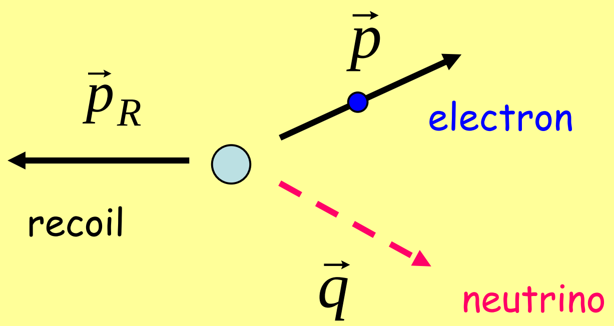
However, in the case of the free neutron, there are no complicated nuclear structure effects, and so the matrix element is identically 1:

$$\lambda_{if} = \frac{2\pi}{\hbar} \frac{G^2}{V^2} \rho_f$$

When this occurs in a nucleus, the beta decay rate is the fastest possible, and the transition is classified as "superallowed"

Density of states factor:

Just like the calculation we did for electron scattering, but now there are two light particles in the final state!



We want to work out the number of equivalent final states within energy interval dE_f of E_f.

$$\rho_f \equiv \frac{dn}{dE_f}$$

Final state momenta are quantized in volume V *(lecture 6)*

$$dn = dn_e \times dn_\nu = \left(4\pi\ p^2 dp\ \frac{V}{h^3}\right) \times \left(4\pi\ q^2 dq\ \frac{V}{h^3}\right)$$

$$E_f = Mc^2 + mc^2 + K_R + K_e + cq$$

But the nucleon is much heavier than the other particles: $K_R = (p_R)^2 / 2M \cong 0$

$$\Rightarrow \quad dE_f\Big|_{K_e = const} = c\ dq$$

Put this all together for the density of states factor:

$$\rho_f \equiv \frac{dn}{dE_f} = \frac{dn_e \times dn_\nu}{c\,dq} = (4\pi)^2 \frac{V^2 \; p^2 q^2}{h^6 c} dp$$

And finally, for the transition rate:

$$\lambda_{if} = G^2 \frac{2\pi}{\hbar c} \left| M_{nuclear} \right|^2 \frac{(4\pi)^2}{h^6} p^2 q^2 \; dp$$

free neutron: $M_{nuclear} = 1$

mixed transition: $G^2 = G_V^2 + 3\,G_A^2$

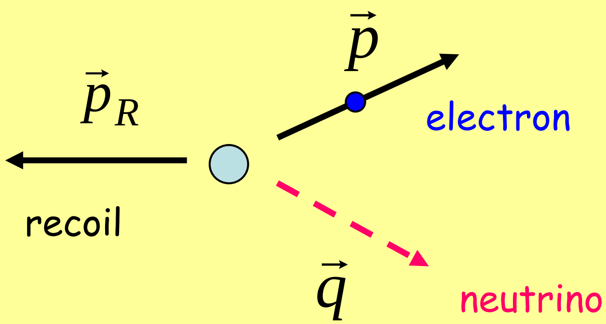
Exercise: plug in all the units and check that the transition rate is in sec^{-1}

Notice: This is actually a partial decay rate, because the electron momentum p is specified explicitly. λ_{if} here gives the rate at which the decay occurs for a given electron momentum falling within dp of p $\rightarrow$ this predicts the momentum spectrum!

Electron momentum spectrum:

$$N(p)\,dp \;=\; N_o\,\lambda_{if} \;=\; (const.)\times p^2 q^2\,dp$$

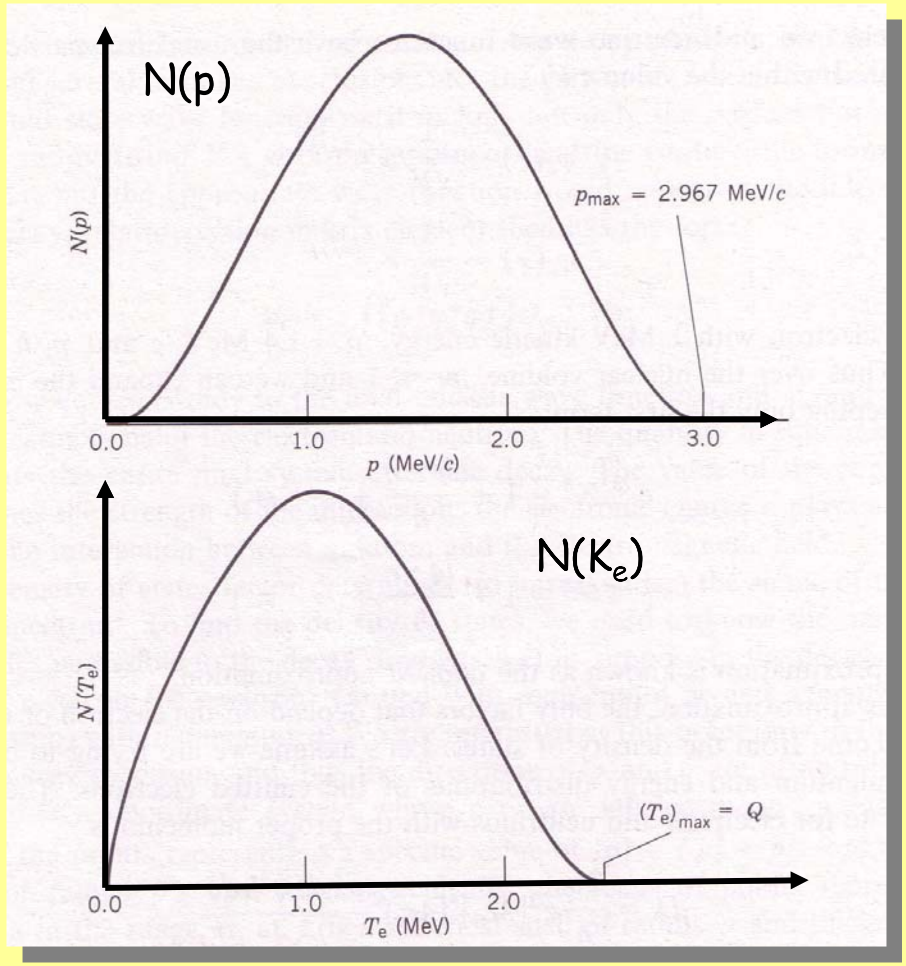
$$\Rightarrow\quad N(p) \;=\; (const.)\times p^2 q^2 \;=\; (const)\;\times p^2\,\overbrace{(Q-K_e)}^{\text{approx: } K_R\,=\,0}{}^2$$



Predicted spectral shapes,
Krane, figure 9.2:

(plotted for Q = 2.5 MeV,
not the neutron!)

(Note: max. K(e^-) = Q)

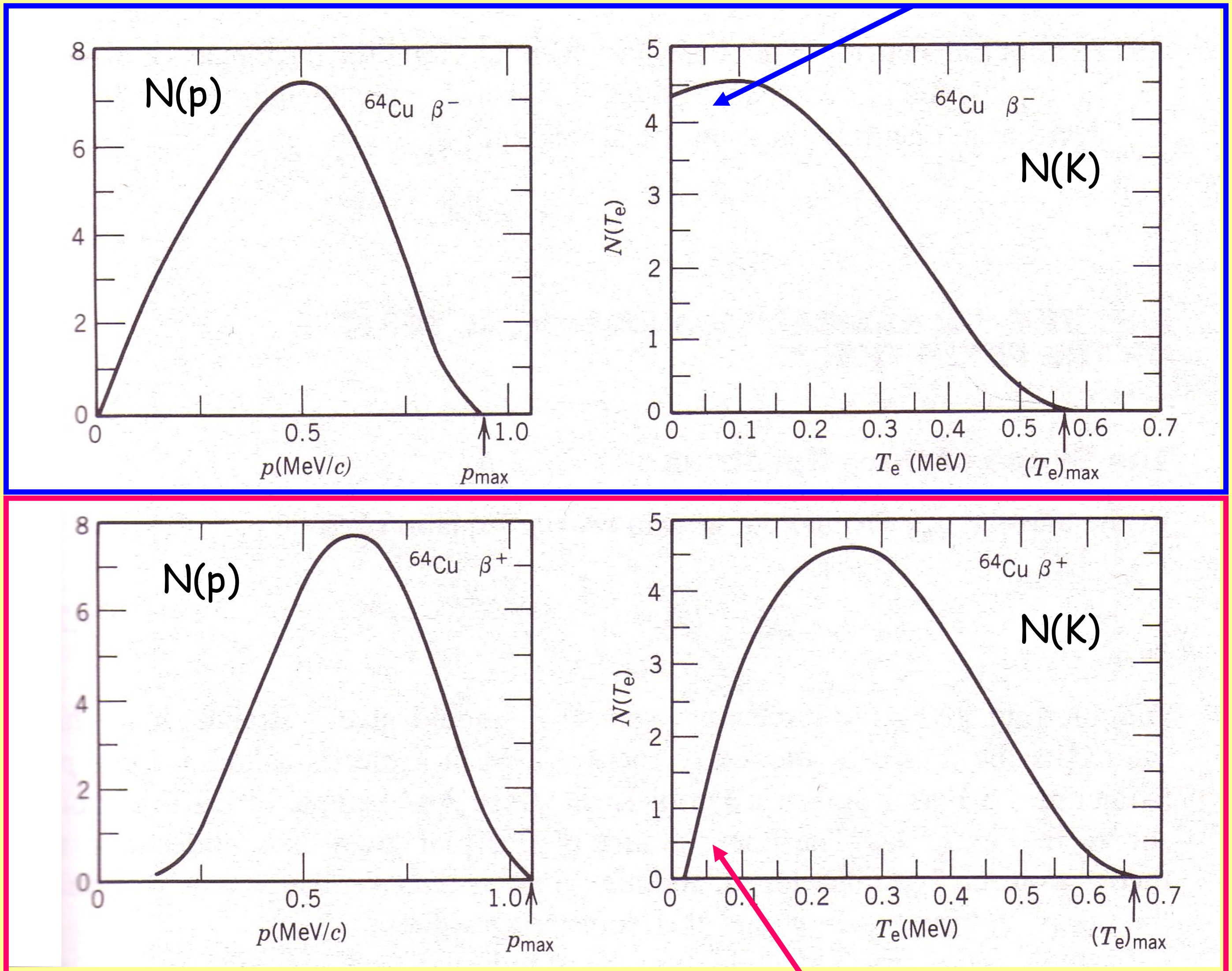


Comparison to Reality: (Krane, fig. 9.3: e+ and e- decays of ^{64}Cu)

???

Coulomb effects …

Too many low energy e^-



Too few low energy e^+

Discrepancy: neglect of Coulomb effects in the final state.

Key point: Coulomb distortions of the energy spectra occur AFTER the electron/positron are emitted in the weak decay process.

Modified density of electron/positron states:

$$dn_e = \left(4\pi \; p^2 dp \, \frac{V}{h^3} \right) \boxed{F(Z', p)}$$

"**Fermi function**", depends on the charge Z′ of the "daughter nucleus" (final state) and the electron/positron momentum

original result

Approximate correction factor for β± decay:

$$F^{\pm}(Z', p) \;\cong\; \frac{x}{1 - e^{-x}} \;, \quad x = \mp \frac{2\pi \,\alpha\, Z'}{\beta}, \quad \beta = \frac{\mathrm{v}}{c}, \quad \alpha = \frac{e^2}{4\pi\varepsilon_o \hbar c} = \frac{1}{137}$$

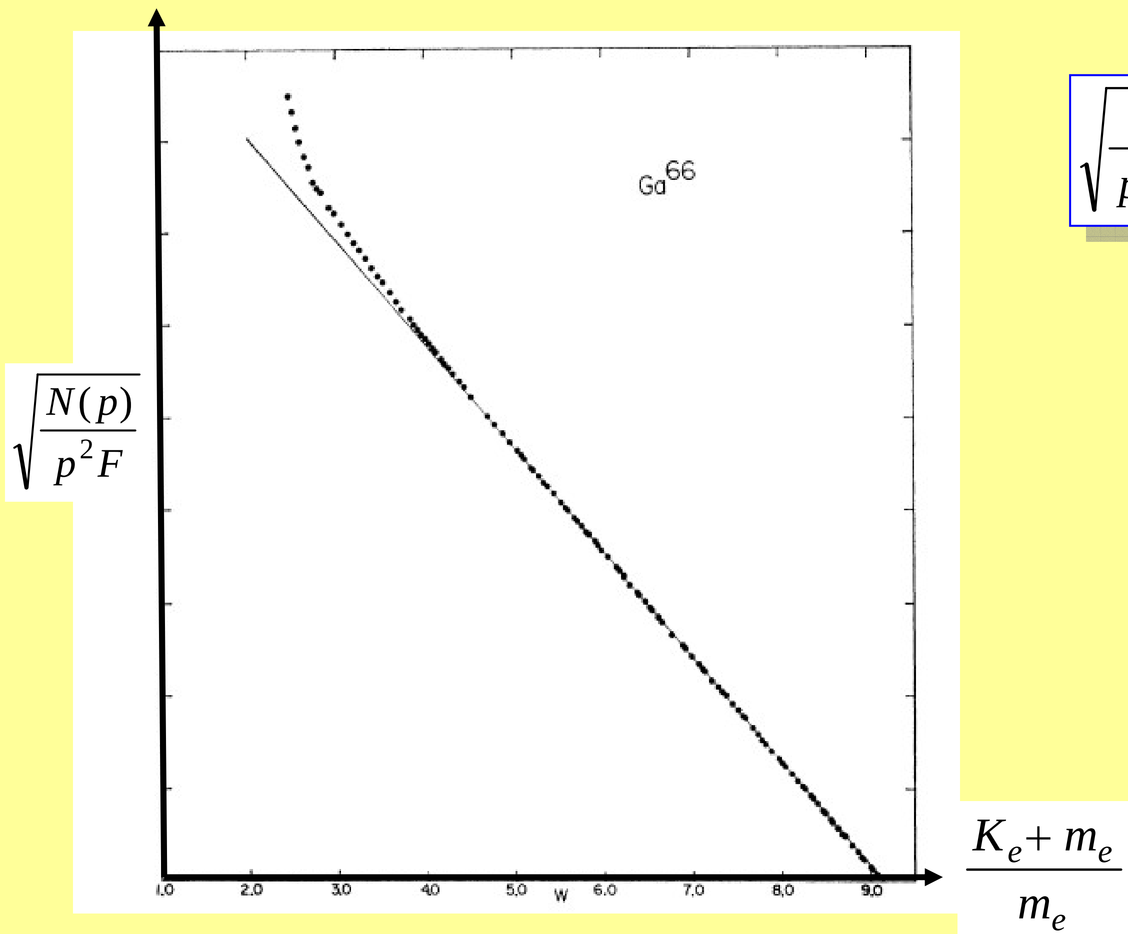
Modified electron/positron spectrum prediction:

$$N(p) = C\; p^2 \; (Q - K_e)^2 \;\; F^{\pm}(Z', p)\,, \quad C = \frac{G^2}{2\pi^3 \hbar^7 c^3} \; \left| M_{nucl} \right|^2$$

Fermi-Kurie Plot

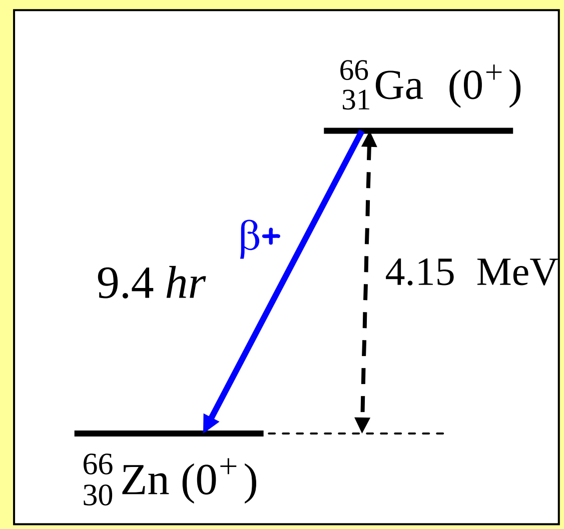
Idea: for "allowed decays", corresponding to our approximation: $e^{i\vec{p}_R \bullet \vec{r}/\hbar} = 1$

inside the nucleus, the electron energy spectrum can be "linearized" if one accounts for the Coulomb distortion via the Fermi function F(Z',p):



$$\sqrt{\frac{N(p)}{p^2 F(Z',p)}} \sim Q - K_e$$

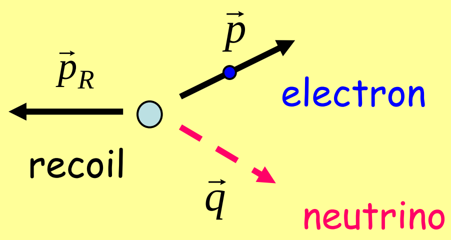
linear function, endpoint Q



^{66}Ga → ^{66}Zn decay, Phys. Rev. ***129****, 1782*

Neutrino Mass effect: $^3\mathrm{H} \rightarrow {}^3\mathrm{He} + e^- + \bar{\nu}_e \quad (Q = 18.6 \;\; \mathrm{keV})$

Idea: shape of the electron energy spectrum near the endpoint (Q) is sensitive to the mass of the electron antineutrino:



recall: $Q \equiv K_R + K_e + K_\nu$

When $K_e \cong Q$, $K_R \cong K_\nu \rightarrow 0$. if $m_\nu \neq 0$, then in this limit, mass effects are most pronounced.

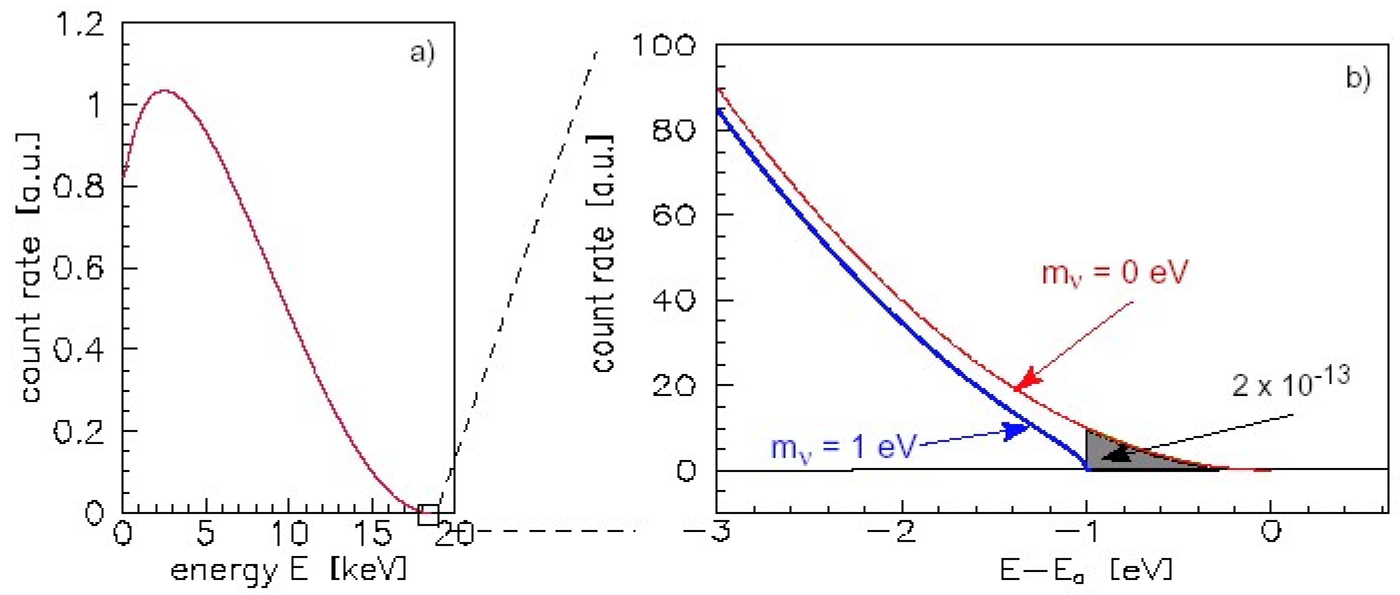


Figure 2: The electron energy spectrum of tritium β decay: (a) complete and (b) narrow region around endpoint E_0. The β spectrum is shown for neutrino masses of 0 and 1 eV.

figure from “KATRIN” proposal, 2001: (Karlsruhe Tritium Neutrino expt.)

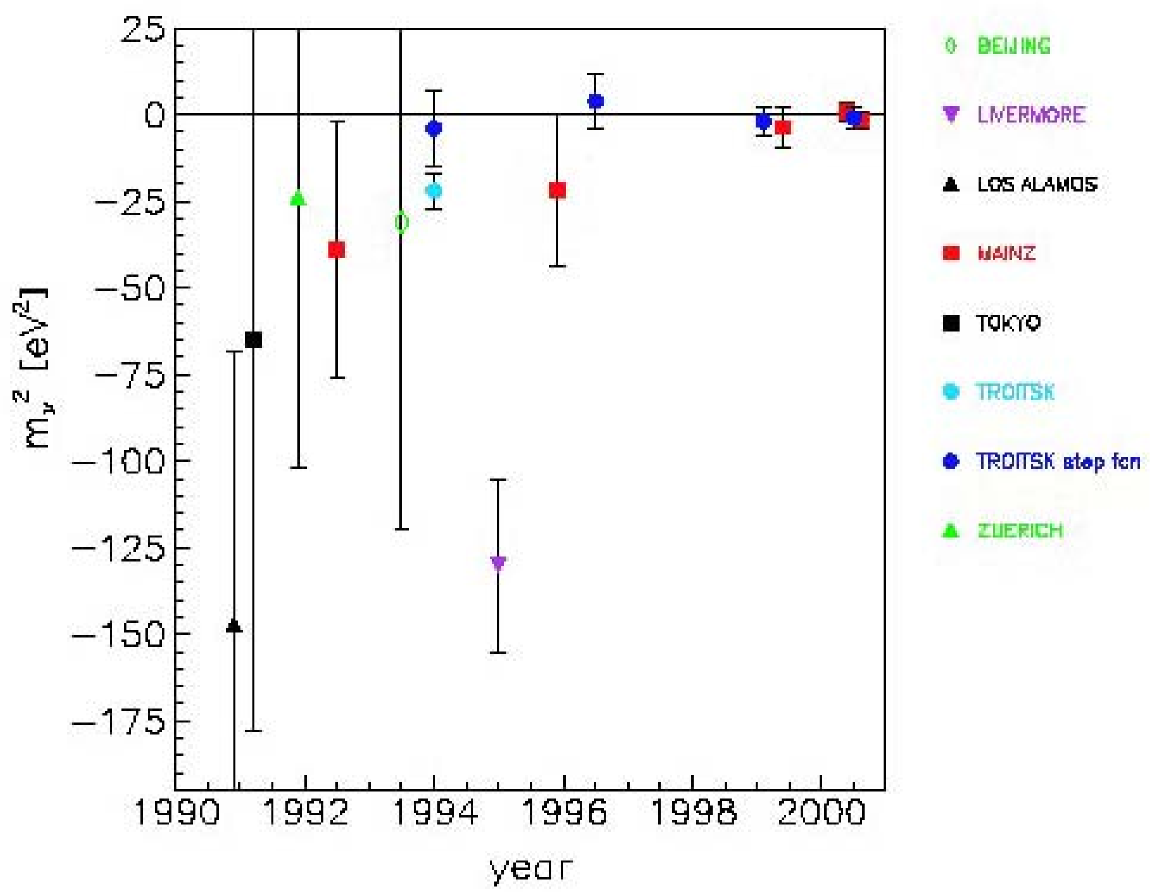


Figure 4: Results of tritium β decay experiments on the observable m_ν^2 over the last decade.

- Best direct upper limit: $m_\nu <$ 2.2 eV
- from Sudbury neutrino observatory and other experiments, we have convincing indirect evidence of nonzero neutrino mass that is much smaller than this

Web sites: http://cupp.oulu.fi/neutrino/nd-mass.html, http://www-ik.fzk.de/~katrin/